TYPE II BLOW UP FOR THE ENERGY SUPERCRITICAL NLS

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Abstract. We consider the energy super critical nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + u|u|^{p-1} = 0$$

in large dimensions $d \geq 11$ with spherically symmetric data. For all $p > p(d)$ large enough, in particular in the super critical regime

$$s_c = \frac{d}{2} - \frac{2}{p-1} > 1,$$

we construct a family of $\mathcal{C}^\infty$ finite time blow up solutions which become singular via concentration of a universal profile

$$u(t,x) \sim \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q\left(\frac{r}{\lambda(t)}\right) e^{i\gamma(t)}$$

with the so called type II quantized blow up rates:

$$\lambda(t) \sim c_u (T-t)^{\frac{\ell}{\alpha}}, \quad \ell \in \mathbb{N}^*, \quad 2\ell > \alpha = \alpha(d,p).$$

The essential feature of these solutions is that all norms below scaling remain bounded

$$\limsup_{t \uparrow T} \|\nabla^s u(t)\|_{L^2} < +\infty \quad \text{for} \quad 0 \leq s < s_c.$$

Our analysis fully revisits the construction of type II blow up solutions for the corresponding heat equation in [15], [34], which was done using maximum principle techniques following [26]. Instead we develop a robust energy method, in continuation of the works in the energy critical case [38], [31], [39], [40] and the L^2 critical case [22]. This shades a new light on the essential role played by the solitary wave and its tail in the type II blow up mechanism, and the universality of the corresponding singularity formation in *both* energy critical and super critical regimes.

1. Introduction

1.1. **The NLS problem.** In this paper we study the focusing nonlinear Schrödinger equation:

$$(NLS) \quad \begin{cases} i\partial_t u + \Delta u + u|u|^{p-1}, \\ u_{|t=0} = u_0 \end{cases} \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \quad u(t,x) \in \mathbb{C}. \tag{1.1}$$

This canonical dissipative model conserves the total energy and mass:

$$E(u(t)) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{p+1}\int |u|^{p+1} = E(u_0), \tag{1.2}$$

$$\int |u(t)|^2 = \int |u_0|^2. \tag{1.3}$$

The scaling symmetry $u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$ for $\lambda > 0$ is an isometry of the homogeneous Sobolev critical space

$$\|u_\lambda(t,\cdot)\|_{\dot{H}^{s_c}} = \|u(\lambda^2 t,\cdot)\|_{\dot{H}^{s_c}} \quad \text{for} \quad s_c = \frac{d}{2} - \frac{2}{p-1}.$$