GLOBAL WELL-POSEDNESS, SCATTERING AND BLOW-UP FOR THE ENERGY-CRITICAL, FOCUSING, NON-LINEAR SCHRÖDINGER EQUATION IN THE RADIAL CASE

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1. Introduction

In this paper, we consider the $\dot{H}^1$ critical non-linear Schrodinger equation

$$\begin{cases} i\partial_t u + \Delta u \pm |u|^{\frac{4}{N-2}} u = 0 & (x,t) \in \mathbb{R}^N \times \mathbb{R} \\ u|_{t=0} = u_0 \in \dot{H}^1(\mathbb{R}^N). \end{cases}$$

Here the $-$ sign corresponds to the defocusing problem, while the $+$ sign corresponds to the focusing problem. The theory of the Cauchy problem (CP) for this equation was developed in [8] (Cazenave and Weissler). They show that if $\|u_0\|_{\dot{H}^1} \leq \delta$, δ small, there exists a unique solution $u \in C(\mathbb{R}; \dot{H}^1(\mathbb{R}^N))$ with the norm $\|u\|_{L_{x,t}^{\frac{2(N+2)}{N-2}}} < \infty$(i.e. the solution scatters in $\dot{H}^1(\mathbb{R}^N)$). See section 2 of this paper for a review of these results.

In the defocusing case, Bourgain ([5], [6]) proved that, for $N = 3, 4$ and u_0 radial, this also holds for $\|u_0\|_{\dot{H}^1} < +\infty$, and that for more regular u_0, the solution preserves the smoothness for all time. (Another proof of this last fact is due to Grillakis [13] for $N = 3$). Bourgain's result was then extended to $N \geq 5$ by Tao [26], still under the assumption that u_0 is radial. Then in [9] (Colliander, Keel, Staffilani, Takaoka and Tao) the result was obtained for general u_0, when $N = 3$. This was extended to $N = 4$ in [24] (Ryckman, Visan) and finally to $N \geq 5$ in [28] (Visan).

In the focusing case, these results do not hold. In fact, the classical virial identity (see for example Glassey in [12] and section 5)

$$\frac{d^2}{dt^2} \int |x|^2 |u_0(x,t)|^2 \, dx = 8 \left\{ \int |\nabla u(t)|^2 - |u(t)|^{\frac{2N}{N-2}} \right\}$$

shows that if $E(u_0) = \frac{1}{2} \int |\nabla u(t)|^2 - \frac{N-2}{2N} \int |u(t)|^{\frac{2N}{N-2}} < 0$ and $|x|\, u_0 \in L^2(\mathbb{R}^N)$, the solution must break down in finite time. Moreover,

$$W(x) = W(x,t) = \frac{1}{\left(1 + \frac{|x|^2}{N(N-2)}\right)^{N-2/2}}$$

is in $\dot{H}^1(\mathbb{R}^N)$ and solves the elliptic equation

$$\Delta W + |W|^{\frac{4}{N-2}} W = 0,$$

so that scattering cannot always occur even for global (in time) solutions.

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